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COMMON FIXED POINT THEOREMS FOR ϕ_p OPERATOR IN CONE BANACH SPACES

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ABSTRACT
In this paper, we generalize and prove common fixed point theorems for ϕ_p Operator in Cone Banach Space

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I. INTRODUCTION
Fixed point theorems are irrevocable in the theory of non linear analysis. In this direction one of the initial and crucial results is the Banach contraction mapping principle [1].Banach (1922) proved that every contraction in complete metric space has a unique fixed point. After this pivotal result, theory of fixed point theorems has been studied by many authors in many directions. In some papers, authors define new contractions and discuss the existence uniqueness of few points for such mappings. Instead of metric spaces, some authors investigate fixed point theorems on various spaces, such as cone metric spaces, metric space, quasi-metric spaces, hyper convex spaces etc.
It is quite natural to consider generalization of the notion to metric $d: X \times X \to [0, \infty)$. The question was, what must [0, ∞) be replaced by. In 1980 Bogdan Rzepecki[2] , in 1987 Shy-Der lin[3] and in 2007 Huang and Zhang [4] gave the same answer: Replace the real number with Banach space ordered by a cone, resulting in the so called cone metric. They also discussed some properties of convergence of sequence and proved the fixed point theorems of contractive mapping for cone metric spaces. Many results on fixed point theorems have been extended to cone metric spaces in [9, 10]. Recently, E. Karpinar [5], presented some fixed point theorems for self mappings satisfying some contractive condition on a cone Banach space. Thobet Abdelijawad, E. Karpinar, and Kenan Tas [6] have given some generalizations to this theorem. Neeraj Malviya and Sarla Chouhan [7] extended some point theorems to cone Banach space. Ali Mutlu and Normin Yolcu [8] proved the fixed point theorems ϕ_p operator for cone Banach space for self map. More precisely, they proved that for a closed and convex sub set C of a cone Banach space with the norm $\|X\|_C = d(x, 0)$. If there exist a, b, c, r and $T: C \to C$ satisfies the conditions $0 \le \phi_q(r) + \phi_q(a) + 2[\phi_q(b) + \phi_q(c)]$ and

$$a\phi_p[d(Tx, Ty)] + b\phi_p[d(x, Tx)] + c\phi_p[d(y, Ty)] \le r\phi_p[d(x, y)]$$

For all $x, y \in C$, then T has at least one fixed point.
Rahul Tiwari and D. P. Shukla[11] , obtained coincidence points and common fixed points in cone Banach spaces which is generalizes and extends the results of [6].
The purpose of this paper is to generalize, extend and improves the results of [8].

II. PRELIMINARY NOTES
Definition 2.1 [4] Let $(E, \|.\|)$ be a real Banach space. A sub set P of E is said to be a cone if and only if ,
(i) P is closed, non empty and $P \neq \{0\}$,
(ii) $a, b \in R,\ a, b \geq 0$ and $x, y \in P \Rightarrow ax + by \in P$,
(iii) $P \cap (-P) = \{0\}$.
For a given cone $P \dot{\subseteq} E$, we define a partial ordering $\le$ with respect to P by $x \le y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate that $x \le y$ but $x \neq y$ while $x \ll y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of P.
The cone P is called normal if there is a number $k > 0$ such that

$$0 \le x \le y \Rightarrow \|x\| \le K\|y\|, \text{ for all } x, y \in E.$$

The least positive number satisfying the above is called the normal constant.
Definition 2.2[4]: Let X be a non empty set. Suppose that, the mapping $d: X \times X \to E$ is said to be cone metric on X, if it is satisfies the following conditions:
(i) $0 \le d(x, y)$ for all $x, y \in X$;
(ii) $d(x, y) = 0$ if and only if $x = y$;
(iii) $d(x, y) = (y, x)$ for all $x, y \in X$;
(iv) $d(x, y) \le d(x, z) + d(z, y)$ for all $x, y, z \in X$.

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International Journal OF Engineering Sciences & Management Research

Then the pair $d(X, d)$ is called cone metric spaces. It is quite natural to consider cone normed spaces (CNS).
Definition 2.3[5]: Let X be a vector space over R. Suppose the mapping $\|.\|_C: X \to E$ satisfies
(i) $\|X\|_C \geq 0$ for $x \in X$,
(ii)) $\|X\|_C = 0$,
(iii) $\|x + y\|_C \leq \|x\|_C + \|y\|_C$ for all $x, y \in X$
(iv) $\|Kx\|_C = |K|\|x\|_C$.
Then $\|.\|_C$ is called a cone norm on X and the pair $(X, \|.\|_C)$ is called a cone normed space. Here we observed that every CNS is CMS of course $d(x, y) = \|x - y\|_C$.
Definition 2.4[5]: Let $(X, \|.\|_C)$ be a cone normed space, $x \in X$ and $\{x_n\}_{\geq 1}$be a sequence in X. Then,
(i) $\{x_n\}_{\geq 1}$ Converges to x whenever $c \in E$ with $0 \ll c$, there is a natural number N such that $\|x_n - x\|_C \ll c$ for all $n \geq N$. It is denoted by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$;
(ii) $\{x_n\}_{\geq 1}$ is a Cauchy sequence whenever for every $c \in E$ with $0 \ll c$, there is a natural number N, such that $\|x_n - x\|_C \ll c$ for all $n, m \geq N$;
(iii) Let $(X, \|.\|_C)$ is a complete cone normed space if every Cauchy sequence is convergent.
Note: The complete cone metric spaces will be called cone Banach space.
Lemma 2.5[4]: Let $(X, \|.\|_C)$ be a cone normed space,P be a normal cone with normal constant K and let $\{x_n\}$ be a sequence in X. Then,
(i) the sequence $\{x_n\}$ converges to x if and only if $\|x_n - x\|_C \to 0 \; as \; n \to \infty$,
(ii) the sequence $\{x_n\}$ Cauchy if and only if $\|x_n - x_m\|_C \to 0 \; as \; n, m \to \infty$,
(iii) the sequence $\{x_n\}$ converges to x and the sequence $\{y_n\}$ converges to$y$, then

$$\|x_n - y_n\|_C \to \text{i} \; \|x - y\|_C.$$

Definition 2.6[5]: Let C be a closed and convex sub set of a cone Banach space with the norm $\|.\|_C$ and $T: C \to C$ be a mapping satisfying the condition

$$\|Tx - Ty\|_C \leq \|x - y\|_C, \forall\, x, y \in C \; \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Then T is called non expansive if it satisfies the condition (1).
Definition 2.7[8]: Let E be Banach algebra and $(E, \|.\|_C)$ be a Banach space. And $\phi_p: E \to E$ is an increasing and positive mapping, *i.e.*

$$\phi_p(x) = \|x\|^{p-2}x, where \; \frac{1}{p}+\frac{1}{q} = 1$$

If $E = R$. Then$\phi_p: R \to R$ is a p −Laplacian operator $i.e.$ $\phi_p(x) = |x|^{p-2}x$ for some $p > 1$.
By using this definition, the operator $\phi_p: E \to E$ holds the following holds:
(1) if $x \leq y$, then$\phi_p(x) \leq \phi_p(y)$ for all $x, y \in E$,
(ii) ϕ_p is a continuous bijection and its inverse mapping is also continuous,
(iii) $\phi_p(xy) = \phi_p(x)\phi_p(y)$ for all $x, y \in E$,
(iv) $\phi_p(x + y) \leq \phi_p(x) + \phi_p(y)$ for all $x, y \in E$.

III. MAIN RESULTS

The results which will give are generalization of the theorems [6] and [7] of [8].
Theorem 3.1: Let C be a closed and convex sub set of a cone Banach space X with the norm $\|.\|_C$. Let E be a Banach algebra and $\phi_p: E \to E$ and $T_1, T_2: C \to C$ be any two mapping which satisfies the condition:

$$\phi_p[d(x, T_1x)] + \phi_p[d(y, T_2y)] \leq K\phi_P[d(x, y)] \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

For all $x, y \in C$, where $2^{p-1} \leq k < 4^{p-1}$ in E. Then T_1 and T_2 have at least common fixed point.
Proof: Let $x_0 \in C$ be arbitrary. We define a sequence $\{x_{2n}\}$ in the following way:

$$x_{2n+1} = \frac{x_{2n+T_1x_{2n}}}{2}, n = 0,1,2 \ldots\ldots\ldots\ldots\ldots (3)$$

And

$$x_{2n+2} = \frac{x_{2n+1+T_2x_{2n+1}}}{2}, n = 0,1,2 \ldots\ldots\ldots\ldots\ldots (4)$$

Then we notice that

$$x_{2n} - T_1x_{2n} = 2\{\, x_{2n} - \frac{x_{2n+T_1x_{2n}}}{2} \}$$
$$= 2\{x_{2n} - x_{2n+1}\} \ldots\ldots\ldots\ldots\ldots (5)$$

And

$$x_{2n+1-T_2x_{2n+1}} = 2\,\{\, x_{2n+1} - \frac{x_{2n+1+T_2x_{2n+1}}}{2} \}$$

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International Journal OF Engineering Sciences & Management Research

$$= \;2\{\, x_{2n+1} - \;x_{2n+2}\,\} \ldots\ldots\ldots\ldots\ldots\ldots \quad (6)$$

This yields that

$$d(x_{2n}, T_1x_{2n}) \;= \|x_{2n} - T_1x_{2n}\|_C$$
$$= 2\|x_{2n} - x_{2n+1}\|_C$$
$$=2d(x_{2n,}x_{2n+1}) \ldots\ldots\ldots\ldots\ldots\ldots \;(7)$$

And

$$d(x_{2n+1}, T_2x_{2n+1}) \;= \|x_{2n+1} - T_2x_{2n+1}\|_C$$
$$= 2\|x_{2n+1} - x_{2n+2}\|_C$$
$$= 2d(x_{2n+1,}x_{2n+2}) \ldots\ldots\ldots\ldots\ldots\ldots \;(8)$$

Substitute $x = x_{2n-1}$ and $y = x_{2n}$ in (2).Then we get

$$\phi_p[d(x_{2n-1}, T_1x_{2n-1})] + \phi_p[d(x_{2n}, T_1x_{2n})] \;\le k\phi_p[d(x_{2n-1}, x_{2n})]$$

By (7), we can obtain

$$\phi_p[2d(x_{2n-1}, x_{2n})] + \phi_p[2d(x_{2n}, x_{2n+1})] \;\le k\phi_p[d(x_{2n-1}, x_{2n})]$$

From the property of ϕ_p operator,

$$\phi_p 2[d(x_{2n-1}, x_{2n}) + d(x_{2n}, x_{2n+1})] \le \; k\phi_p[d(x_{2n-1}, x_{2n})]$$

When the essential arrangement is applied, we can get

$$d(x_{2n}, x_{2n+1}) \le \left(\frac{\phi_q(k)}{2} - 1\right) d(x_{2n-1}, x_{2n}).$$

Repeating this relation

$$d(x_{2n}, x_{2n+1}) \le \left(\frac{\phi_q(k)}{2} - 1\right) d(x_0, x_1) \ldots\ldots\ldots\ldots\ldots\ldots (9)$$

Let $m > n$, then from (9), we have

$$d\big(x_{2m,}\, x_{2n}\big) \le d(x_{2m}, x_{2m-1}) + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots + d(x_{2n+1}, x_{2n})$$
$$\le \left[\left(\frac{\phi_q(k)}{2} - 1\right)^{2m-1} + \cdots \ldots\ldots\ldots\ldots\ldots\ldots\ldots.. + \left(\frac{\phi_q(k)}{2} - 1\right)^{2n}\right] d(x_1, x_0)$$
$$\le \frac{\left(\frac{\phi_q(k)}{2} - 1\right)}{2 - \frac{\phi_q(k)}{2}}\; d(x_1, x_0).$$

Since $2^{p-1} \le k < 4^{p-1}$, $\{x_{2n}\}$ is a Cauchy sequence in C. Because C is a closed and convex sub set of a cone Banach space, thus $\{x_{2n}\}$ sequence converges to some $x^* \in C$.That is

$x_{2n} \to \;\; x^*, x^* \;\in C.$

Regarding the inequality

$$d(x^*, T_1x_{2n}) \le d(x^*, x_{2n}) + d(x_{2n}, T_1x_{2n})$$

From (7),

$$d(x^*, T_1x_{2n}) \le d(x^*, x_{2n}) + 2d(x_{2n}, x_{2n+1}) \text{ as} n \to \infty.$$

then $\;d(x^*, T_1x_{2n}) = 0$.Thus, $T_1x_{2n} \to x^*$.

Finally, we substitute $x = x^*$ and $y = x_{2n}$ in (2). Then we get

$$\phi_p[d(x^*, T_1x^*)] + \phi_p[d(x_{2n}, T_1x_{2n})] \;\le k\phi_p[d(x^*, x_{2n})]$$

From the property of ϕ_p mapping,

$$\phi_p[d(x^*, T_1x^*) + d(x_{2n}, T_1x_{2n})] \;\le k\phi_p[d(x^*, x_{2n})]$$

By using (7), we obtain

$d(x^*, x_{2n}) + 2d(x_{2n}, x_{2n+1}) \le \phi_q(k)$. When $n \to \infty$, $d(x^*, T_1x^*) = 0$. Then $x^* = \; T_1x^*$. Hence $\;x^*\;$ has at least a fixed point of $T_{1.}$.

Similarly, we can show that $T_2x^* = \; x^*$.Thus $T_1x^* = \; x^* = T_2x^*$.

Hence $\;x^*$ is common fixed point of $T_1$and T_2.This completes the proof of the theorem.

Theorem 3.2: Let C be a closed and convex sub set of a cone Banach space X with the norm $\|.\|_C$. Let E be a Banach algebra and $\phi_p: E \to E$ and $T_1, T_2: C \to C$ be any two mapping which satisfies the condition:

$a\phi_p[d(T_1x, T_2y)] + b\phi_p[d(x, T_1x)] + c\phi_p[d(y, T_2y)] \le r\phi_p[d(x, y)]$ for all $x, y \in C$, where $0 \;\le\; \phi_q < \phi_q(a) + 2\big[\phi_q(b) + \phi_q(c)\big]$ .Then T_1 and T_2 have at least common fixed point.

Proof: Construct a sequence $\{x_{2n}\}$as in proof of theorem 3.1. Then

$$x_{2n} - T_1x_{2n-1} \;= 1/2(\,x_{2n-1} - T_1x_{2n-1})$$

Thus;

$$d(x_{2n} - T_1x_{2n-1}\,) = 1/2d(\,x_{2n-1} - T_1x_{2n-1}) \ldots\ldots\ldots\ldots\ldots\ldots (10)$$

In addition, we know that

$d(x_{2n}, T_1x_{2n}) \;\; = 2d(x_{2n}, x_{2n+1})$.Thus the triangle inequality, implies that

IJESMR
International Journal OF Engineering Sciences & Management Research

$$d(x_{2n},T_1x_{2n}) - d(x_{2n} - T_1x_{2n-1}) \leq d(T_1x_{2n-1},T_2x_{2n})$$

Then from (10) and (7) , we get

$$2d(x_{2n},x_{2n+1}) - d(x_{2n-1} - x_{2n}) \leq d(T_1x_{2n-1},T_2x_{2n})$$

By substituting $x = x_{2n-1}$ and $y = x_{2n}$ in (10)

$$a\phi_p[d(T_1x_{2n-1},T_2x_{2n})] + b\phi_p[d(x_{2n-1},T_1x_{2n-1})] + c\phi_p[d(x_{2n},T_2x_{2n})] \leq r\phi_p[d(x_{2n-1},x_{2n})]$$

As in the proof of theorem (9) we can obtain

$d(x_{2n},x_{2n+1}) \leq \left[\frac{\phi_q(r)+\phi_q(a)-2\phi_q(b)}{2\phi_q(a)+\phi_q(c)}\right] d(x_{2n-1},x_{2n})$ for all $n \geq 1$. Repeating this relation, we get $d(x_{2n},x_{2n+1}) \leq h^n\, d(x_0,x_1)$……………………………………(11)

Where h= $\left[\frac{\phi_q(r)+\phi_q(a)-2\phi_q(b)}{2\phi_q(a)+\phi_q(c)}\right] < 1.$

Let $m > n$, then from (11), we have

$$d(x_{2m},x_{2n}) \leq d(x_{2m},x_{2m-1}) + \cdots \ldots \ldots \ldots \ldots \ldots + d(x_{2n+1},x_{2n})$$
$$\leq [h^{2m-1} + \cdots \ldots \ldots \ldots + h^{2n}]d(x_1,x_0)$$
$$\leq \frac{h^{2n}}{1-h} d(x_1,x_0). \ldots\ldots\ldots (12)$$

Thus $\{x_{2n}\}$ is a Cauchy sequence in C and thus it is converges to some $x^* \in C$. As in the proof of theorem 3.1, we can show $T_1x^* = x^*$ and also we can show $T_1x^* = x^* = T_2x^*$.

Thus, x^*has at least one common fixed point of $T_1$and T_2.This completes the proof of the theorem.

Theorem 3.3: Let C be a closed and convex sub set of a cone Banach space X with the norm $\|.\|_C$. Let E be a Banach algebra and $\phi_p: E \to E$ and $T_1, T_2: C \to C$ be any two mapping which satisfies the condition:

$a\phi_p[d(T_1x,T_2y)] + b\phi_p[d(x,T_2y)] + c\phi_p[d(y,T_1x)] \leq r\phi_p[d(x,y)]$ for all $x, y \in C$, where $0 \leq \phi_q < \phi_q(a) + 2[\phi_q(b) + \phi_q(c)]$ .Then T_1 and T_2 have at least common fixed point.

Proof: Construct a sequence $\{x_{2n}\}$as in proof of theorem 3.1. Then

$$x_{2n} - T_1x_{2n-1} = 1/2(x_{2n-1} - T_1x_{2n-1})$$

Thus;

$$d(x_{2n} - T_1x_{2n-1}) = 1/2d(x_{2n-1} - T_1x_{2n-1}) \ldots\ldots\ldots (13)$$

In addition, we know that

$d(x_{2n},T_1x_{2n}) = 2d(x_{2n},x_{2n+1})$.Thus the triangle inequality, implies that

$$d(x_{2n},T_1x_{2n}) - d(x_{2n} - T_1x_{2n-1}) \leq d(T_1x_{2n-1},T_2x_{2n})$$

Then from (10) and (7) , we get

$$2d(x_{2n},x_{2n+1}) - d(x_{2n-1} - x_{2n}) \leq d(T_1x_{2n-1},T_2x_{2n})$$

By substituting $x = x_{2n-1}$ and $y = x_{2n}$ in (13)

$$a\phi_p[d(T_1x_{2n-1},T_2x_{2n})] + b\phi_p[d(x_{2n-1},T_2x_{2n})] + c\phi_p[d(x_{2n},T_1x_{2n-1})] \leq r\phi_p[d(x_{2n-1},x_{2n})]$$

As in the proof of theorem (9) we can obtain

$d(x_{2n},x_{2n+1}) \leq \left[\frac{\phi_q(r)+\phi_q(a)-2\phi_q(b)}{2\phi_q(a)+\phi_q(c)}\right] d(x_{2n-1},x_{2n})$ for all $n \geq 1$. Repeating this relation, we get $d(x_{2n},x_{2n+1}) \leq h^n\, d(x_0,x_1)$……………………………………(11)

Where h = $\left[\frac{\phi_q(r)+\phi_q(a)-2\phi_q(b)}{2\phi_q(a)+\phi_q(c)}\right] < 1.$

Let $m > n$, then from (11), we have

$$d(x_{2m},x_{2n}) \leq d(x_{2m},x_{2m-1}) + \cdots \ldots \ldots \ldots \ldots \ldots + d(x_{2n+1},x_{2n})$$
$$\leq [h^{2m-1} + \cdots \ldots \ldots \ldots + h^{2n}]d(x_1,x_0)$$
$$\leq \frac{h^{2n}}{1-h} d(x_1,x_0). \ldots\ldots\ldots \quad (12)$$

Thus $\{x_{2n}\}$ is a Cauchy sequence in C and thus it is converges to some $x^* \in C$.

The rest of the proof is similar to the proof of the theorem 3.2.

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IJESMR
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